Precalculus

The Language of Everything That Follows

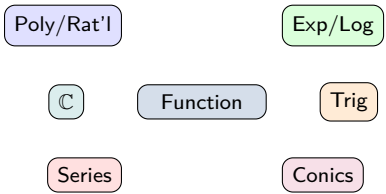
GRE Mathematics — Physics First Series

Chapter 1

Where this chapter sits

Precalculus is the **grammar** of the whole exam. Six families of functions, two bridges between them.

- Functions: domain, range, inverse
- Polynomial & rational
- Exponential & logarithm
- Trigonometry (unit circle)
- Analytic geometry (conics)
- Sequences, series, complex numbers



Objectives

You will be able to:

1. state a function's domain/range and decide invertibility;
2. find polynomial roots and read a rational graph's poles vs. holes;
3. use the log laws and solve exponential equations;
4. evaluate trig from the unit circle and wield the core identities;
5. classify a conic by completing the square;
6. sum series and test geometric convergence;
7. compute in $\mathbb{C}$ with Euler's formula and De Moivre.

Functions are rule *plus* domain

Priming Question — answer before the next slide

For $f(x) = \sqrt{x}$ and $g(x) = x^2$, for which x does $f(g(x)) = x$ actually hold?

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Only $x \geq 0$: $f(g(x)) = \sqrt{x^2} = |x| = x$ *only* when $x \geq 0$.

- Invertible $\iff$ bijective; inverse is reflection across $y = x$.
- A domain restriction can turn a non-invertible function invertible.

Worked Example 1 — composition with a domain

Let $f(x) = \dfrac{1}{x-2}$, $g(x) = \sqrt{x}$. Find $f \circ g$ and its domain.

$$(f \circ g)(x) = \frac{1}{\sqrt{x}-2}.$$

Domain needs $x \geq 0$ (for g) **and** $\sqrt{x} \neq 2 \Rightarrow x \neq 4$.

Key Result

$\mathrm{dom}(f \circ g) = [0,4) \cup (4,\infty)$ — the excluded point is the exam's trap.

Roots run polynomials; degrees run rational graphs

- Degree $n \Rightarrow n$ roots in $\mathbb{C}$; real coefficients $\Rightarrow$ conjugate pairs.
- Rational $r = p/q$: zeros from p, vertical asymptotes from q, end behavior from comparing degrees; a **cancelled** factor is a hole, not an asymptote.

Key Result

$\deg p < \deg q \Rightarrow y = 0$; $\deg p = \deg q \Rightarrow y = \frac{\text{lead } p}{\text{lead } q}$; $\deg p = \deg q + 1 \Rightarrow$ slant asymptote.

Exponential $\leftrightarrow$ logarithm: one idea, two sides

Priming Question — answer before the next slide

Why is log the "right" inverse of b^x — what problem does it uniquely solve?

Exponential $\leftrightarrow$ logarithm: one idea, two sides

Priming Question — answer before the next slide

Why is log the "right" inverse of b^x — what problem does it uniquely solve?

It isolates the *exponent*: $b^x = y \iff x = \log_b y$.

Key Result

$$\log_b(xy) = \log_b x + \log_b y, \quad \log_b(x^r) = r \log_b x, \quad \log_b x = \frac{\ln x}{\ln b}.$$

But $\log(x + y) \neq \log x + \log y$ — the sum law is for products only.

Worked Example 2 — a hidden quadratic

Solve $e^{2x} - 5e^x + 6 = 0$. Let $u = e^x > 0$:

$$u^2 - 5u + 6 = (u-2)(u-3) = 0 \Rightarrow u = 2,\ 3.$$

Both positive, so both are valid: $x = \ln 2$ or $x = \ln 3$.

Key Result

Substitution $u = e^x$ turns an exponential equation into algebra — but keep only $u > 0$ roots, since $e^x > 0$ always.

Trigonometry: memorize parents, breed the rest

One picture — the unit circle point $(\cos\theta, \sin\theta)$ — defines all six functions, their signs, and their periods.

Key Result

$\sin^2\theta + \cos^2\theta = 1$; $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$; $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$.

Divide the Pythagorean identity by $\cos^2$ or $\sin^2$ to regenerate the others; the angle-sum formulas breed the double- and half-angle formulas.

Conics, series, and the complex plane

- **Conics:** complete the square to reach standard form; the discriminant $B^2 - 4AC$ classifies (ellipse < 0, parabola $= 0$, hyperbola > 0).
- **Geometric series:** $\displaystyle\sum_{n\geq 0} ar^n = \frac{a}{1-r}$ iff $|r| < 1$ (terms$\to 0$ is necessary, not sufficient — see the harmonic series).
- **Complex plane:** $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$; multiplying by i *rotates* 90°; De Moivre powers and roots.

Key Result

Euler's formula is the seam where exponential, trigonometric, and complex all meet.

Synthesis — what to carry into Chapter 2

- Function = rule + domain; invertible $\iff$ bijective.
- Roots and degrees decide polynomial/rational behavior; holes $\neq$ asymptotes.
- Every log law is an exponent law reversed; solve exponentials by taking a log.
- Own the Pythagorean and angle-sum identities; derive the rest.
- Complete the square for any conic; geometric series converge iff $|r| < 1$.
- $\mathrm{e}^{\mathrm{i}\theta}$ unifies growth, rotation, and oscillation.

30 practice problems ($\star$ to $\star\star\star$) follow in the manuscript, §Practice Problems.