Calculus I

The derivative and its uses

one limit
the derivative

Lesson `gre.calc1.ch2`

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One limit, a whole toolkit

Differential calculus is the study of ***instantaneous rate of change****, and all of it grows from a single limit — the derivative* $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$*: the slope of the tangent, the velocity of a moving point, the sensitivity of an output to its input.*

limits • the derivative • rules & chain • optima & related rates • sketching • the integral & FTC

Establish the limit once; a short table of rules then differentiates anything.

Limits: what the outputs approach

❓ $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$ — *substitution gives* $\frac{0}{0}$. *Now what?*

Factor and cancel the vanishing factor, then substitute again.

$$\frac{(x-3)(x+3)}{x-3} = x + 3 \;\Rightarrow\; 6$$

The limit is the value the outputs approach — here a removable hole at $(3, 6)$, *whether or not the function is defined there.*

The derivative: secant $\to$ tangent

*The difference quotient is the slope of a **secant** line; as $h \to 0$ the second point slides in and the secant pivots into the **tangent**. Its slope is the derivative.*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For $f(x) = x^2$: the quotient is $2x + h$, and letting $h \to 0$ gives $f'(x) = 2x$.

Worked example — a derivative from the definition

Find $f'(x)$ for $f(x) = x^2$ from first principles.

$$
\begin{aligned}
(x+h)^2 - x^2 &= 2xh + h^2 && \text{expand} \\
\frac{2xh + h^2}{h} &= 2x + h && \text{cancel } h \\
\lim_{h \to 0}(2x + h) &= 2x && \text{let } h \to 0
\end{aligned}
$$

$$f'(x) = 2x$$

*Used once, the definition yields the general **rules** — after which we differentiate by pattern.*

The differentiation rules

Power, constant-multiple, and sum rules handle the simple cases; the product and quotient rules handle the two ways functions combine that addition cannot.

product & quotient

$$(uv)' = u'v + uv' \qquad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

"First times derivative of second, plus second times derivative of first" — and "bottom d-top minus top d-bottom, over bottom squared."

Worked example — the product rule

Differentiate $f(x) = x^2 e^x$.

$$
\begin{aligned}
u = x^2,\ v = e^x \qquad u' = 2x,\ v' = e^x && \text{identify} \\
f' = u'v + uv' = 2x\,e^x + x^2 e^x && \text{apply} \\
= e^x(x^2 + 2x) && \text{factor}
\end{aligned}
$$

$$
f'(x) = e^x(x^2 + 2x)
$$

Factor the common e^x; at $x = 0$ the derivative is 0, a flat minimum.

The chain rule: the most-used rule

? *How do you differentiate a composition $y = f(g(x))$?*

Derivative of the outside, leaving the inside alone, times the derivative of the inside — because a sensitivity is passed along a linkage by multiplication.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(g(x)) \cdot g'(x)$$

Watch out

Almost every real derivative is a chain-rule problem — and forgetting the inner $g'(x)$ is the #1 error.

Worked example — the chain rule

Differentiate $y = (3x^2 + 1)^5$.

$$\begin{aligned} \text{inner } u &= 3x^2 + 1, \;\; u' = 6x && \text{name inside} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 5(3x^2 + 1)^4 \cdot 6x && \text{outer} \times \text{inner} \\ &= 30x(3x^2 + 1)^4 \end{aligned}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 30x(3x^2 + 1)^4$$

The lone x *out front is the fingerprint of the inside* $6x$.

Implicit differentiation

When a relation is not solved for y, differentiate both sides in x and attach $\frac{\mathrm{d}y}{\mathrm{d}x}$ to every y-term by the chain rule, then solve for y'.

$$x^2 + y^2 = 25 \;\Rightarrow\; 2x + 2y\,y' = 0 \;\Rightarrow\; y' = -\frac{x}{y}$$

At $(3, 4)$ the slope is $-\frac{3}{4}$, perpendicular to the radius — as a tangent to a circle must be.

Related rates and optimization

Two pure applications. ***Related rates:*** *differentiate the equation linking the quantities with respect to time, and substitute the given values last.* ***Optimization:*** *write the target in one variable, set its derivative to zero.*

$$\text{ladder: } 2xx' + 2yy' = 0 \Rightarrow \boxed{y' = -1.5 \text{ ft/s}}$$

Fence-against-a-wall with 100 *m: the max-area pen is* 25×50 *— half the fence on the long side.*

What f' and f'' reveal

f' gives slope: positive rises, negative falls, zero is a candidate peak/valley. f'' gives concavity: positive cups up, negative caps down, a switch is an inflection.

second-derivative test

$$f'(c) = 0: \quad f''(c) > 0 \Rightarrow \text{min}, \quad f''(c) < 0 \Rightarrow \text{max}$$

For $x^3 - 3x$: min at $(1, -2)$, max at $(-1, 2)$, inflection at $x = 0$.

The integral and the Fundamental Theorem

An antiderivative undoes the derivative; the definite integral is signed area. The Fundamental Theorem says these are the same thing.

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) \quad (F' = f)$$

$$\int_0^2 (3x^2 + 2x)\,\mathrm{d}x = [x^3 + x^2]_0^2 = 12$$

Accumulated area and undone slope are literally the same function.

Carry these into Chapter 3

One limit builds the derivative. The chain rule is king. Implicit differentiation attaches y' to every y-term. f' and f'' sketch the curve. The Fundamental Theorem ties differentiation to area — the doorway to integration in Chapter 3.

30 practice problems ($\star$ to $\star\star\star$) in the manuscript.

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