Calculus II

Integration techniques, applications, and infinite series

integrate
& sum

Lesson `gre.calc2.ch3`

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The three movements of Calculus II

Chapter 2 built the derivative; Chapter 3 does the reverse and its consequences. Three movements: ***techniques*** *for evaluating integrals,* ***applications*** *that turn integrals into areas, volumes, and lengths, and* ***infinite series*** *— adding up infinitely many terms and asking when the sum is finite.*

integration techniques • applications of the integral • series & convergence

Power series tie the two halves together: a function becomes an infinite polynomial.

Integration techniques: pick the tool by the form

Substitution undoes the chain rule. Parts undoes the product rule. Trig integrals and trig substitution tame roots and powers of sine/cosine. Partial fractions split a rational function into pieces you already know how to integrate.

by parts

$$\int u\,\mathrm{d}v \;=\; uv \;-\; \int v\,\mathrm{d}u \qquad \text{(LIATE picks } u\text{)}$$

Match the integrand's shape to the method — that recognition is the whole game.

Worked example — integration by parts

Evaluate $\int x\,\mathrm{e}^{x}\,\mathrm{d}x$.

$$u = x,\ \mathrm{d}v = \mathrm{e}^{x}\mathrm{d}x \Rightarrow \mathrm{d}u = \mathrm{d}x,\ v = \mathrm{e}^{x} \qquad \text{LIATE}$$

$$\int x\mathrm{e}^{x}\mathrm{d}x = x\mathrm{e}^{x} - \int \mathrm{e}^{x}\,\mathrm{d}x \qquad uv - \int v\,\mathrm{d}u$$

$$= x\mathrm{e}^{x} - \mathrm{e}^{x} + C$$

$$\int x\mathrm{e}^{x}\mathrm{d}x = \mathrm{e}^{x}(x-1) + C$$

Differentiate to check: $\mathrm{e}^{x}(x-1) + \mathrm{e}^{x} = x\mathrm{e}^{x}$, *the integrand.*

Worked example — partial fractions

Evaluate $\displaystyle\int \frac{1}{x^2-1}\,\mathrm{d}x$.

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}$$

$$\int = \tfrac{1}{2}\ln|x-1| - \tfrac{1}{2}\ln|x+1| + C$$

$$= \tfrac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

Split the denominator's factors, solve for the constants, integrate each piece as a log.

Improper integrals: integrate to a limit

❓ *What does* $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ *even mean?*

Replace the infinite bound with b and take a limit: $\int_1^b x^{-2}\mathrm{d}x = 1 - \frac{1}{b} \to 1$ *as* $b \to \infty$.

$$\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = 1 \quad \text{(converges)}$$

The p-integral $\int_1^\infty x^{-p}\mathrm{d}x$ *converges exactly when* $p > 1$.

Applications: the integral as an accumulator

Area between curves, volumes of revolution (disks/washers, or cylindrical shells), arc length, and average value are all one idea: slice, approximate each slice, integrate.

volumes

disk/washer: $\int \pi\left(R^2 - r^2\right)\mathrm{d}x$; shell: $\int 2\pi\,(\text{radius})(\text{height})\,\mathrm{d}x$

Choose disks or shells by which one makes the slice perpendicular vs. parallel to the axis.

Worked example — a volume of revolution

Rotate $y = \sqrt{x}$, $0 \le x \le 4$, about the x-axis. Find the volume.

$$\begin{aligned} V &= \int_0^4 \pi\left(\sqrt{x}\right)^2 \,\mathrm{d}x \qquad \text{disk: } R = \sqrt{x} \\ &= \pi \int_0^4 x \,\mathrm{d}x = \pi \left[\tfrac{x^2}{2}\right]_0^4 \\ &= \pi \cdot 8 \end{aligned}$$

$$V = 8\pi$$

Each slice is a disk of radius $\sqrt{x}$; sum their areas πR^2 along the axis.

Sequences: does the list settle?

A sequence is a function on the positive integers. It converges if its terms approach a single limit. A bounded, monotone sequence must converge — a fact the series tests lean on.

$$a_n = \frac{3n}{n+1} \to 3 \quad \text{as } n \to \infty$$

Divide by the highest power of n to read the limit at infinity.

Series: when does an infinite sum stay finite?

❓ $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ *converges, but* $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ *diverges. Why?*

Rate. A geometric series converges iff $|r| < 1$*; a p-series iff* $p > 1$*. The* ***ratio test****, comparison, integral, and alternating-series tests decide the rest.*

⚠ Necessary, not sufficient

Terms $\to 0$ does not guarantee convergence — the harmonic series proves it.

Worked example — the ratio test

Test $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$.

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2}{n+1}$$

$$\lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$$

$$\text{ratio} = 0 < 1 \Rightarrow \text{converges}$$

When the limit ratio is below one, the terms shrink geometrically fast — convergence.

Power series: a function as an infinite polynomial

A power series $\sum c_n(x-a)^n$ converges on an interval set by its radius (find it with the ratio test). A ***Taylor series*** *chooses $c_n = \dfrac{f^{(n)}(a)}{n!}$ so the series is the function near a.*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Memorize the expansions of e^x, $\sin x$, $\cos x$, and $\dfrac{1}{1-x}$ — most others build from them.

Parametric and polar calculus

In polar coordinates a region's area is $\frac{1}{2}\int r^2\,\mathrm{d}\theta$*, and a parametric curve's arc length is* $\int \sqrt{(x')^2+(y')^2}\,\mathrm{d}t$*. Same integrals, new coordinates.*

$$\text{area swept} = \frac{1}{2}\int_{\alpha}^{\beta} r(\theta)^2\,\mathrm{d}\theta$$

Polar area sums thin circular sectors, not rectangles.

Carry these into Chapter 4

Match each integral to its technique. The integral accumulates area, volume, and length. A series converges by rate: geometric iff $|r| < 1$, p-series iff $p > 1$, else use the ratio/comparison/integral/alternating tests. Taylor series turn functions into polynomials.

30 practice problems ($\star$ to $\star\star\star$) in the manuscript.

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