# Linear Algebra

Vectors, matrices, and the maps between spaces

**one map**
a matrix

Lesson `gre.linalg.ch5`

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# One idea: the matrix is a linear map

*A matrix is not a grid of numbers — it is a **linear transformation** written in coordinates. Everything in the chapter is one of three questions about that map: does it have a solution, does it have an inverse, and what does it do to special directions.*

systems & elimination • determinants & inverses • vector spaces & basis
rank–nullity • linear transformations • eigenvalues • orthogonality

*The determinant tests invertibility; the eigenvectors are the directions the map only stretches.*

# Systems: Gaussian elimination to echelon form

*Solving $Ax = b$ means row-reducing the augmented matrix. Row operations never change the solution set, so reduce until each pivot is alone in its column, then read off the answer.*

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 11 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

*A zero row equal to a nonzero constant means no solution; a free column means infinitely many.*

# Worked example — a matrix product is rows times columns

*Compute $AB$ with $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.*

$$(1,1) = 1 \cdot 5 + 2 \cdot 7 = 19$$
$$(1,2) = 1 \cdot 6 + 2 \cdot 8 = 22$$
$$(2,1) = 3 \cdot 5 + 4 \cdot 7 = 43, \quad (2,2) = 50$$

$$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

*Order matters: $BA \neq AB$ in general — composing maps depends on which you apply first.*

# Determinants & inverses: one number decides

**?** *When does a square matrix have an inverse?*

*Exactly when its determinant is nonzero. The determinant measures how the map scales area/volume; a zero determinant collapses space, so the map cannot be undone.*

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Watch out**

$\det(A + B) \neq \det A + \det B$, but $\det(AB) = \det A \det B$.

# Worked example — a $2 \times 2$ inverse

*Invert* $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$.

$$\det A = 2 \cdot 3 - 1 \cdot 5 = 1 \qquad \text{nonzero}$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

*Swap the diagonal, negate the off-diagonal, divide by the determinant. Check:* $AA^{-1} = I$.

# Vector spaces: independence, basis, dimension

*A **basis** is a smallest spanning set: enough vectors to build everything, few enough that none is redundant. The number of them is the **dimension**.*

**independence test**

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = 0 \text{ forces all } c_i = 0$$

*Vectors are dependent exactly when one is a combination of the others — when the matrix of them has a nonpivot column.*

# The rank–nullity theorem

*Every column of $A$ is either a pivot (it adds a dimension to the output) or free (it adds a dimension of solutions to $A\mathbf{x} = 0$). Counting both ways gives one clean law.*

$$\underbrace{\operatorname{rank} A}_{\text{pivots}} + \underbrace{\operatorname{nullity} A}_{\text{free vars}} = n \quad \text{(columns)}$$

*Rank is the dimension of the column space; nullity is the dimension of the null space.*

# Linear transformations wear a matrix

*Any linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is "multiply by a matrix." Its columns are just the images of the basis vectors — $T(\mathrm{e}_j)$ is the $j$th column. Composition of maps is multiplication of their matrices.*

$$T(\mathrm{x}) = A\mathrm{x}, \qquad (S \circ T) \leftrightarrow BA$$

*Change of basis conjugates the matrix: $A' = P^{-1}AP$.*

# Eigenvalues: the directions a map only stretches

*Which vectors keep their direction under $A$?*

*The eigenvectors: $A\mathbf{v} = \lambda\mathbf{v}$. Find the stretch factors $\lambda$ from the characteristic equation, then solve for each eigenvector.*

$$\det(A - \lambda I) = 0$$

*If $A$ has $n$ independent eigenvectors it diagonalizes: $A = PDP^{-1}$, which makes powers trivial, $A^k = PD^kP^{-1}$.*

# Worked example — eigenvalues of a $2 \times 2$

*Find the eigenvalues of* $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

$$\begin{aligned} \det(A - \lambda I) &= (2 - \lambda)^2 - 1 = 0 \\ &= \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) \end{aligned}$$

$$\lambda = 1 \text{ and } 3$$

*Eigenvalues sum to the trace* $(1 + 3 = 4 = 2 + 2)$ *and multiply to the determinant* $(1 \cdot 3 = 3)$ *— a free check.*

# Orthogonality, projection, Gram–Schmidt

*Orthogonal vectors have zero dot product. Projecting* $\mathrm{b}$ *onto* $\mathrm{a}$ *keeps only the part along* $\mathrm{a}$*;* ***Gram–Schmidt*** *turns any basis into a perpendicular one by subtracting off those projections.*

$$\operatorname{proj}_{\mathrm{a}} \mathrm{b} = \frac{\mathrm{a} \cdot \mathrm{b}}{\mathrm{a} \cdot \mathrm{a}} \, \mathrm{a}$$

*An orthogonal matrix* ($Q^{\top} Q = I$) *preserves lengths and angles — a rotation or reflection.*

# Carry these into the exam

*A matrix is a linear map. Elimination solves systems and reveals rank. A nonzero determinant means invertible. Rank + nullity = number of columns. Eigenvectors are the unstretched directions;* $\det(A - \lambda I) = 0$ *finds their stretch factors; n independent eigenvectors diagonalize. Orthogonal matrices preserve geometry.*

30 practice problems ($\star$ to $\star\star\star$) in the manuscript.

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