Differential Equations

Equations whose unknown is a function

find

$y(x)$

Lesson `gre.odes.ch4`

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An equation whose unknown is a function

*A **differential equation** relates a function to its own derivatives. Solving one means finding the function $y(x)$ that fits, not a number. The **order** is the highest derivative present; an equation is **linear** if y and its derivatives appear only to the first power and are never multiplied together.*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky \quad \bullet \quad y' + P(x)y = Q(x) \quad \bullet \quad y'' + by' + cy = g(x)$$

A general solution carries arbitrary constants; an initial condition pins them down.

First-order: separable equations

? *When can you just integrate both sides?*

When the equation separates: get all the y's (with $\mathrm{d}y$) on one side and all the x's (with $\mathrm{d}x$) on the other, then integrate each side.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y) \;\Rightarrow\; \int \frac{\mathrm{d}y}{h(y)} = \int g(x)\,\mathrm{d}x$$

Don't forget the single $+C$ — it is what an initial condition later fixes.

Worked example — a separable IVP

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2xy$, *with* $y(0) = 3$.

$$\begin{aligned}
\int \frac{\mathrm{d}y}{y} &= \int 2x\,\mathrm{d}x && \text{separate} \\
\ln|y| &= x^2 + C && \text{integrate} \\
y &= A\,\mathrm{e}^{x^2} && A = \mathrm{e}^{C} \\
y(0) = 3 &\Rightarrow A = 3 && \text{apply IC}
\end{aligned}$$

$$y = 3\,\mathrm{e}^{x^2}$$

Apply the initial condition after integrating, never before.

First-order linear: the integrating factor

A first-order linear equation $y' + P(x)y = Q(x)$ is solved by multiplying through by $\mu = \mathrm{e}^{\int P\,\mathrm{d}x}$, chosen so the left side becomes the derivative of a product.

integrating factor

$$\mu = \mathrm{e}^{\int P\,\mathrm{d}x}, \quad \text{then } (\mu y)' = \mu Q, \quad y = \frac{1}{\mu}\int \mu Q\,\mathrm{d}x$$

The whole trick: μ turns the left side into one clean derivative you can undo.

Worked example — integrating factor

Solve $y' + \dfrac{1}{x}\,y = x$ for $x > 0$.

$$\mu = e^{\int (1/x)\,dx} = e^{\ln x} = x \qquad P = 1/x$$

$$(xy)' = x \cdot x = x^2 \qquad \text{multiply by } \mu$$

$$xy = \tfrac{1}{3}x^3 + C \qquad \text{integrate}$$

$$y = \tfrac{1}{3}x^2 + \tfrac{C}{x} \qquad \text{divide by } x$$

$$y = \tfrac{1}{3}x^2 + \frac{C}{x}$$

Recognizing the left side as $(\mu y)'$ is the entire method.

Exact equations and substitutions

*$M\,\mathrm{d}x + N\,\mathrm{d}y = 0$ is **exact** when $M_y = N_x$; then it is $\mathrm{d}F = 0$ for some potential F, and the solution is $F(x, y) = C$. When an equation is neither separable nor linear, a substitution (homogeneous $v = y/x$, or Bernoulli $v = y^{1-n}$) can reduce it to one that is.*

Test first

Check $M_y = N_x$ before hunting for F — if it fails, look for an integrating factor or a substitution instead.

Modeling: growth, decay, and cooling

The same first-order template describes a lot of the world. "Rate proportional to amount" gives exponential growth or decay; Newton's cooling drives temperature toward the ambient value.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \qquad \Rightarrow \; P = P_0 \mathrm{e}^{kt}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_a) \qquad \Rightarrow \; T = T_a + (T_0 - T_a)\mathrm{e}^{-kt}$$

Sign of k decides growth vs decay; cooling always relaxes toward T_a.

Second-order linear: try $y = e^{rx}$

? *Why does e^{rx} solve $ay'' + by' + cy = 0$?*

*Because differentiating e^{rx} just multiplies by r, substitution collapses the ODE to an algebra problem — the **characteristic equation**.*

$$ay'' + by' + cy = 0 \;\Rightarrow\; ar^2 + br + c = 0$$

Solve the quadratic in r; its roots decide the shape of the solution.

The three root cases

Everything hinges on the discriminant of $ar^2 + br + c = 0$.

Roots	General solution
real, distinct $r_1 \neq r_2$	$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
real, repeated r	$y = (C_1 + C_2 x) e^{rx}$
complex $a \pm bi$	$y = e^{ax}(C_1 \cos bx + C_2 \sin bx)$

Complex roots give oscillation — Euler's formula turns $e^{(a \pm bi)x}$ into decaying sines and cosines.

Worked example — complex roots

Solve $y'' + 4y' + 13y = 0$.

$$r^2 + 4r + 13 = 0 \qquad \text{characteristic}$$

$$r = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3\mathrm{i} \qquad \text{complex}$$

$$y = \mathrm{e}^{-2x}(C_1 \cos 3x + C_2 \sin 3x) \qquad a = -2,\ b = 3$$

$$y = \mathrm{e}^{-2x}(C_1 \cos 3x + C_2 \sin 3x)$$

Real part of the root sets the decay; imaginary part sets the frequency.

Nonhomogeneous: particular + homogeneous

For $ay'' + by' + cy = g(x)$, the full solution is $y = y_h + y_p$: the homogeneous solution y_h (from the characteristic equation) plus one particular solution y_p that produces $g(x)$.

undetermined coefficients

guess y_p shaped like $g(x)$ (polynomial, e^{kx}, sin / cos); match coefficients

If your guess already solves the homogeneous equation, multiply it by x (resonance).

Applications: oscillators and circuits

The constant-coefficient second-order equation is the language of vibration and circuits. A mass on a spring and an RLC circuit obey the same ODE.

$$mx'' + cx' + kx = 0 \qquad \bullet \qquad Lq'' + Rq' + \tfrac{1}{C}q = 0$$

Underdamped systems (c small) give a decaying sinusoid — exactly the complex-root case.

What to carry away

Classify first: order and linearity. First order — try separable, then integrating factor, then exact/substitution. Model "rate proportional to amount" as exponential. Second order constant-coefficient — write the characteristic equation and read the root case. Nonhomogeneous — add a particular solution to the homogeneous one.

30 practice problems ($\star$ to $\star\star\star$) in the manuscript.

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