Precalculus

The language of everything that follows

one idea
the function

Lesson `gre.precalc.ch1`

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One idea, six families

*Precalculus is the **grammar** of the whole Subject Test. Everything is a **function** — a rule with a domain — and each family below is just a class of functions distinguished by how it is built and how it grows.*

polynomial/rational • exponential/log • trigonometric
analytic geometry • sequences & series • complex numbers

Two bridges tie them together: the logarithm inverts the exponential, and Euler's formula links trigonometry to the complex plane.

A function is rule *plus* domain

? *For $f(x) = \sqrt{x}$ and $g(x) = x^2$, for which x does $f(g(x)) = x$?*

Compose carefully: $f(g(x)) = \sqrt{x^2} = |x|$, which equals x only when $x \geq 0$.

$$f(g(x)) = |x| = x \iff x \geq 0$$

Invertible means one-to-one; the inverse reflects the graph across $y = x$. A domain restriction can create an inverse.

Worked example — track the hidden exclusion

$f(x) = \dfrac{1}{x-2}$, $g(x) = \sqrt{x}$. Find $f \circ g$ and its domain.

$$(f \circ g)(x) = \frac{1}{\sqrt{x} - 2} \qquad \text{substitute } g \text{ into } f$$

$$\text{need} \ : \ x \geq 0 \qquad \text{from } g$$

$$\text{and} \ : \ \sqrt{x} \neq 2 \Rightarrow x \neq 4 \qquad \text{from } f$$

$$\text{domain} = [0, 4) \cup (4, \infty)$$

The excluded point $x = 4$ is invisible in the formula — you find it by tracking the inner function into the outer one.

Roots run polynomials; degrees run rational graphs

Degree n gives n complex roots; real coefficients force conjugate pairs. For a rational $r = p/q$: zeros from p, vertical asymptotes from q — but a cancelled factor is a hole, not an asymptote.

end behavior

$$\deg p < \deg q : y = 0; \quad \deg p = \deg q : y = \frac{\text{lead } p}{\text{lead } q}; \quad \deg p = \deg q + 1 : \text{slant}$$

Always reduce to lowest terms before naming asymptotes.

Exponential $\leftrightarrow$ logarithm: one idea, two sides

? *Why is* log *the "right" inverse of* b^x*?*

It isolates the exponent: $b^x = y \iff x = \log_b y$*. Every log law is an exponent law read backwards.*

$$\log_b(xy) = \log_b x + \log_b y, \quad \log_b(x^r) = r \log_b x$$

Watch out

$\log(x + y) \neq \log x + \log y$ — the sum law is for *products* only.

Worked example — an equation that is secretly quadratic

Solve $e^{2x} - 5e^{x} + 6 = 0$.

$$
\begin{aligned}
u = e^{x} > 0 \qquad & \text{substitute} \\
u^2 - 5u + 6 = (u-2)(u-3) = 0 \qquad & \text{factor} \\
u = 2 \text{ or } 3 \qquad & \text{both } > 0 \ \checkmark
\end{aligned}
$$

$$
x = \ln 2 \text{ or } \ln 3
$$

Keep only positive u: $e^{x} > 0$ always. A negative root would be rejected.

Trigonometry: memorize parents, breed the rest

One picture — the unit-circle point $(\cos\theta, \sin\theta)$ *— defines all six functions, their signs, and their periods.*

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

Divide the Pythagorean identity by $\cos^2$ *or* $\sin^2$*, or set* $\beta = \alpha$*, to regenerate every other identity.*

Worked example — an exact value by subtracting angles

Find $\cos 15^\circ$.

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ) \\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}\end{aligned}$$

$$\cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4} \approx 0.966$$

Split any awkward angle into a sum or difference of $30, 45, 60, 90$.

Conics: complete the square, then read everything

Every "identify this conic" question is a completing-the-square question in disguise. Group, complete each square, match a standard form.

$$x^2 + y^2 - 6x + 4y + 9 = 0 \;\Rightarrow\; (x-3)^2 + (y+2)^2 = 4$$

circle, center $(3, -2)$, radius 2

The discriminant $B^2 - 4AC$ classifies: < 0 ellipse, $= 0$ parabola, > 0 hyperbola.

Sequences & series: convergence is about rate

Arithmetic adds d; geometric multiplies by r. The infinite geometric series settles only when the ratio is small enough.

geometric series

$$\sum_{n\geq 0} ar^n \quad = \quad \frac{a}{1-r} \quad \text{if} \quad |r| \quad < \quad 1$$

Terms $\to 0$ is necessary, not sufficient: the harmonic series $\sum 1/n$ diverges.

Complex numbers: multiplication rotates

? *What does multiplying by* i *do to a point?*

It rotates it 90° *counterclockwise. In polar form* $z = re^{i\theta}$*, multiplication adds arguments and multiplies moduli.*

$$(1+\mathrm{i})^8 = (\sqrt{2})^8\, e^{i\cdot 8\pi/4} = 16\, e^{i2\pi} = \boxed{16}$$

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ *is the seam joining growth, rotation, and oscillation.*

Carry these into Chapter 2

Function = rule + domain. Roots and degrees decide polynomial/rational behavior. Every log law is an exponent law reversed. Own the Pythagorean and angle-sum identities. Complete the square for any conic. Geometric series converge iff $|r| < 1$. $\mathrm{e}^{\mathrm{i}\theta}$ unifies it all.

30 practice problems ($\star$ to $\star\star\star$) in the manuscript.

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