Additional Topics

The final quarter of the exam, seven fields at a glance

seven fields

one pass

Lesson `gre.topics.ch7`

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Seven fields, exam-relevant depth

The last quarter of the Subject Test samples widely. You do not need research depth; you need to recognize the central object of each field and run one standard move.

real analysis • point-set topology • complex analysis
combinatorics & probability • numerical analysis
logic & set theory • algorithms & complexity

Breadth over depth: know the definition, the key theorem, and the one computation each field asks.

Real analysis: the rigor under calculus

? *What does $\lim_{x \to a} f(x) = L$ mean, precisely?*

For every tolerance $\varepsilon > 0$ there is a window $\delta > 0$ so that $0 < |x - a| < \delta$ forces $|f(x) - L| < \varepsilon$. Convergence of a sequence is the same game, indexed by N.

Key theorems on a closed interval: a continuous function attains its max/min (EVT) and hits every intermediate value (IVT). Monotone + bounded $\Rightarrow$ convergent.

Worked example — an ε–δ limit

Prove $\lim_{x\to 3}(2x-1)=5$.

$$
\begin{aligned}
|(2x-1)-5| &= 2|x-3| && \text{simplify the target}\\
2|x-3|<\varepsilon &\iff |x-3|<\tfrac{\varepsilon}{2} && \text{solve for the window}\\
\text{choose } \delta &= \tfrac{\varepsilon}{2} && \text{works for every } \varepsilon>0
\end{aligned}
$$

the limit is 5

$\delta=\varepsilon/2$ mirrors the slope 2: a steeper line needs a tighter input window.

Point-set topology: nearness without distance

*A set is **open** if every point has a little room around it inside the set; **closed** if it contains all its limit points. Continuity: preimages of open sets are open.*

Heine–Borel

in $\mathbb{R}^n$: **compact** $\iff$ closed *and* bounded

Compactness bottles the EVT; connectedness bottles the IVT. Both are how analysis theorems generalize beyond the line.

Complex analysis: one derivative, enormous consequences

? *When is $f(z) = u + \mathrm{i}v$ differentiable as a complex function?*

*Exactly when the **Cauchy–Riemann** equations hold: $u_x = v_y$ and $u_y = -v_x$. Such analytic functions are rigid — knowing them on a boundary fixes them inside.*

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum(\text{residues inside } C)$$

The residue theorem turns hard real integrals into a sum of local data at poles.

Worked example — a residue

Find the residue of $f(z) = \dfrac{e^z}{z^2}$ at $z = 0$ (a double pole).

$$\begin{aligned} \text{Res} &= \lim_{z\to 0} \frac{d}{dz}\left(z^2 \cdot \frac{e^z}{z^2} \right) \\ &= \lim_{z\to 0} \frac{d}{dz}\, e^z = \lim_{z\to 0} e^z \\ \text{residue} &= 1 \end{aligned}$$

For a pole of order m, differentiate $m-1$ times after clearing the pole, then take the limit.

Combinatorics & probability: count, then divide

Order matters → permutations $\frac{n!}{(n-r)!}$*; order doesn't → combinations* $\binom{n}{r}$*.*
Overcounting overlaps? ***Inclusion–exclusion****. Too many pigeons?* ***Pigeonhole****.*

conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad \text{Bayes flips the condition}$$

Expectation is the probability-weighted average outcome, $E[X] = \sum x\, P(x)$*.*

Worked example — inclusion–exclusion

Of 100 *students*, 60 *take French*, 45 *Spanish*, 20 *both. How many take neither?*

$$|F \cup S| = 60 + 45 - 20 = 85$$

$$\text{neither} = 100 - 85$$

15 take neither

Subtract the double-counted overlap once; then take the complement.

Numerical analysis: answers when formulas run out

Newton's method *rides the tangent line to a root:* $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, *roughly doubling correct digits each step.*

Integrals you can't do in closed form: the **trapezoid** *rule (straight tops) and* **Simpson's** *rule (parabolic tops, far more accurate for smooth curves).*

Worked example — one Newton step

Approximate $\sqrt{2}$: solve $f(x) = x^2 - 2 = 0$ from $x_0 = 1.5$.

$$x_1 = 1.5 - \frac{(1.5)^2 - 2}{2(1.5)}$$
$$= 1.5 - \frac{0.25}{3}$$
$$x_1 \approx 1.4167 \quad (\text{true } \sqrt{2} \approx 1.4142)$$

One step already gives three correct digits — quadratic convergence.

Logic & set theory: proof and the sizes of infinity

Negate carefully: "for all x, P" becomes "there exists x with not P." Proof by contradiction, contrapositive, and induction (ordinary and strong) are the workhorses.

Two infinite sets have the same size if a bijection pairs them. $\mathbb{Q}$ is countable; $\mathbb{R}$ is not — Cantor's diagonal argument builds a real no list can contain.

Algorithms & complexity: how cost scales

? *Why is binary search $O(\log n)$?*

Big-O names the dominant growth term, ignoring constants. Each comparison in binary search halves the remaining list, so k steps cover 2^k items — $k = \log_2 n$.

$$O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(2^n)$$

On the exam, count how the work multiplies when the input doubles.

One pass, seven recognitions

Real analysis: ε–δ and IVT/EVT. Topology: open/closed, compact = closed + bounded. Complex: Cauchy–Riemann and residues. Counting: permutations vs. combinations, inclusion–exclusion, Bayes. Numerics: Newton and Simpson. Logic: quantifier negation, induction, countability. Complexity: big-O growth.

30 practice problems ($\star$ to $\star\star\star$) in the manuscript.

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